HW Problem (5) Solve $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$.

[Ans: $\tan y = \frac{1}{2}(x^2-1) + ce^{-x^2}$].

HW (6) Solve $\frac{dz}{dx} + \left(\frac{z}{x}\right)\log z = \frac{z}{x}(\log z)^2$.

[Ans: $(\log z)^{-1} = 1 + cx$].

(7) $\frac{dy}{dx} + y\tan x = y^3 \sec x$.

Definition: A D.E. of the form $M(x,y)dx+N(x,y)dy=0 \rightarrow (1)$ is said to be exact if L.H.S. of (1) is the exact differential of some function $u(x,y)$.

(ie) $du = Mdx + Ndy = 0$.

$\therefore$ Its solution is $u(x,y) = c$.

Theorem: The necessary and sufficient condition for the D.E. $Mdx+Ndy=0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Proof:

Condition is necessary:

The equation $Mdx+Ndy=0$ will be exact, if

$Mdx + Ndy = du \rightarrow (1)$ where u is some function of x and y.

But $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \rightarrow (2)$.

Comparing the coefficients of dx and dy in (1) & (2), we get

$M = \frac{\partial u}{\partial x}$ and $N = \frac{\partial u}{\partial y}$.

Now $\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}$ and $\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$.

But $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

$\therefore \boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ is the necessary condition for exactness.

Condition is sufficient:

(i.e) If $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then $M\,dx + N\,dy = 0$ is exact.

Let $\int M\,dx = u$ [y treated as constant].

Then $\frac{\partial}{\partial x}\left(\int M\,dx\right) = \frac{\partial u}{\partial x} \Rightarrow M = \frac{\partial u}{\partial x} \longrightarrow (3)$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

(or) $\frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$.

$\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ (given) & $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$

(i.e) $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$.

Integrating both sides w.r.t. x (taking y as constant)

$N = \frac{\partial u}{\partial y} + f(y) \longrightarrow (4)$ where $f(y)$ is a function of y alone.

$\therefore M\,dx + N\,dy = \frac{\partial u}{\partial x}dx + \left[\frac{\partial u}{\partial y} + f(y)\right]dy$ (by (3) & (4))

$= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + f(y)\,dy$.

$= du + f(y)\,dy$ using (2).

$= d\left[u + \int f(y)\,dy\right] \longrightarrow (5)$.

which shows that $M\,dx + N\,dy = 0$ is exact.

Method of solution: Using (5), the equation $M\,dx + N\,dy = 0$ becomes $d\left[u + \int f(y)\,dy\right] = 0$.

Integrating $u + \int f(y)\,dy = 0$.

But $u = \int\limits_{y\text{ constant}} M\,dx$ and $f(y)$ = terms of N not containing x.

$\therefore$ The solution is (for $M\,dx + N\,dy = 0$)

$$\int\limits_{y\text{ constant}} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = 0.$$

provided $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Problem ①: Solve $\left(y^2 e^{xy^2} + 4x^3\right)dx + \left(2xye^{xy^2} - 3y^2\right)dy = 0.$

Solution: Equation is $Mdx + Ndy = 0.$

$\therefore M = y^2 e^{xy^2} + 4x^3$ and $N = 2xye^{xy^2} - 3y^2.$

$\therefore \frac{\partial M}{\partial y} = 2ye^{xy^2} + y^2 e^{xy^2} \cdot 2xy;\quad \frac{\partial N}{\partial x} = 2ye^{xy^2} + 2xy \cdot e^{xy^2} \cdot y^2.$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ The equation is exact.

$\therefore$ Solution is $\int\limits_{(y\text{ constant})} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = C.$

(i.e) $\int\limits_{(y\text{ constant})} \left(y^2 e^{xy^2} + 4x^3\right)dx + \int(-3y^2)\,dy = C.$

$\Rightarrow y^2 e^{xy^2} \cdot \frac{1}{y^2} + 4\frac{x^4}{4} - \frac{3y^3}{3} = C.$

$\Rightarrow \boxed{e^{xy^2} + x^4 - y^3 = C}$

Problem ② Solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0.$

Solution: Given equation can be written as

$(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0.$

Here $M = y\cos x + \sin y + y$ and $N = \sin x + x\cos y + x.$

$\Rightarrow \frac{\partial M}{\partial y} = \cos x + \cos y + 1,\quad \frac{\partial N}{\partial x} = \cos x + \cos y + 1.$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ equation is exact.

Its solution is $\int\limits_{(y\text{ const.})} M\,dx + \int (\text{those terms of } N \text{ not containing } x)\,dy = C.$

$\int\limits_{(y\text{ const.})} \left(y\cos x + \sin y + y\right)dx + \int (0)\,dx = C$

(i.e) $\Rightarrow \boxed{y\sin x + (\sin y + y)\,x = C}$

Problem 3

Solve $(a^2 - 2xy - y^2)\,dx - (x+y)^2\,dy = 0$.

Solution: $M = a^2 - 2xy - y^2$; $N = -(x+y)^2$.

$\Rightarrow \frac{\partial M}{\partial y} = -2x - 2y$ & $\frac{\partial N}{\partial x} = -2(x+y)$.

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ equation is exact.

Solution is

$\int_{(y\ \text{const.})} M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$.

$\Rightarrow \int (a^2 - 2xy - y^2)\,dx + \int (-y^2)\,dy = c$.

$\Rightarrow \boxed{a^2x - x^2y - xy^2 - \frac{y^3}{3} = c}$

Problem 4 Solve $(2x^2y + 4x^3 - 12xy^2 + 3y^3 - xe^y + e^{2x})\,dy + (12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y)\,dx = 0$.

Solution: Rearranging the terms, we get

$(2x^2y\,dy + 2xy^2\,dx) + (4x^3dy + 12x^2y\,dx) - (xe^y dy + e^y dx) + 3y^2 dy$
$- (12xy^2 dy + 4y^3)\,dx + 4x^3dx = 0$.
$+ (e^{2x}dy + 2ye^{x}dx)$

$\Rightarrow d(x^2y^2) + d(4x^3y) - d(4xy^3) + d(y^3) - d(xe^y) + d(e^{2x}y) + d(x^4) = 0$.

Integrating we get

$\boxed{x^2y^2 + 4x^3y - 4xy^3 + y^3 - xe^y + e^{2x}y + x^4 = c}$

HW: (5) $(x^2 - ay)\,dx = (ax - y^2)\,dy$.

(6) $ye^{xy}dx + (xe^{xy} + 2y)\,dy = 0$.

16/08/2020 ①

Procedure to solve the exact equation $Mdx+Ndy=0$.

Problem ⑦

Solve $(x^2-2xy+3y^2)dx+(y^2+6xy-x^2)dy=0$.

Solution: The given equation is of the form

$Mdx+Ndy=0$.

$\therefore M = x^2-2xy+3y^2$; $N = y^2+6xy-x^2$

$\Rightarrow \frac{\partial M}{\partial y} = -2x+6y$ $\quad\Rightarrow \frac{\partial N}{\partial x} = 6y-2x$.

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ This is an exact equation

solution is $\int\limits_{(y\,\text{constant})} M\,dx + \int (\text{those terms of } N \text{ not containing } x)dy = c$.

$\Rightarrow \int\limits_{(y\,\text{constant})} (x^2-2xy+3y^2)dx + \int y^2 dy = c$.

$\Rightarrow \frac{x^3}{3} - \frac{2x^2y}{2} + 3xy^2 + \frac{y^3}{3} = c$.

$\Rightarrow \boxed{x^3 - 3x^2y + 9xy^2 + y^3 = c_1}$

Problem ⑧ Test for exactness and solve $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$.

⑨ Solve $(2xy-\sec^2 x)dx + (x^2+2y)dy=0$. $\rightarrow$ Solve.

⑩ $(1+e^x y + xe^x y)dx + (xe^x+2)dy = 0 \rightarrow$ Solve

⑪ $(x^2+2ye^{2x})dy + (2xy+2y^2e^{2x})dx = 0$.

⑫ Solve $(2x+e^x \log y)dx + \left(\frac{e^x}{y}+1\right)dx = 0$.

A differential equation which is not ... sometimes be converted into an exact equation by multiplying it by a suitable factor called an integrating factor (I.F.)

The rules for finding integrating factors of the equation $Mdx + Ndy = 0$ are as follows:

I.F. found by inspection:

Some of the standard results to be used in finding the I.F.

(i) $x\,dy + y\,dx = d(xy)$, (ii) $\frac{x\,dy - y\,dx}{x^2} = d(y/x)$

(iii) $\frac{x\,dy - y\,dx}{xy} = d[\log(y/x)]$ (iv) $\frac{y\,dx - x\,dy}{y^2} = d(x/y)$.

(v) $\frac{x\,dy - y\,dx}{x^2+y^2} = d(\tan^{-1} y/x)$, (vi) $\frac{y\,dx - x\,dy}{x^2+y^2} = d(\tan^{-1}\frac{x}{y})$.

(vii) $\frac{x\,dy - y\,dx}{x^2-y^2} = d\left(\frac{1}{2}\log\frac{x+y}{x-y}\right)$.

Problem ① Solve $y(2xy+e^x)dx = e^x dy$.

$M = 2xy^2+e^x$, $N = e^x$

$\frac{\partial M}{\partial y} = 4xy$ $\quad \frac{\partial N}{\partial x} = e^x$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\Rightarrow$ The given D.E. is not exact

Solution: $y(2xy+e^x)dx = e^x dy$.

$\Rightarrow 2xy^2dx + ye^x dx - e^x dy = 0$

$\Rightarrow (ye^x dx - e^x dy) + 2xy^2 dx = 0$.

So multiplying the above equation by I.F

I.F $= \frac{1}{y^2}$.

we get $\frac{ye^x}{y^2}dx - \frac{e^x}{y^2}dy + 2x\,dx = 0$.

$\Rightarrow \frac{ye^x dx - e^x dy}{y^2} + 2x\,dx = 0 \Rightarrow d\left(\frac{e^x}{y}\right) + 2x\,dx = 0$.

Integrating, we get $\boxed{\frac{e^x}{y} + x^2 = c}$ which is the required solution.

Problem ②: Solve $x\,dx + y\,dy = a(x^2+y^2)\,dy$.

Solution: The given D.E. is

$x\,dx + y\,dy = a(x^2+y^2)\,dy$

$\Rightarrow \dfrac{x\,dx + y\,dy}{x^2+y^2} = a\,dy$.

$\Rightarrow \dfrac{d(x^2+y^2)}{x^2+y^2} = 2a\,dy$.

Integrating $\Rightarrow \log(x^2+y^2) = 2ay + c$. //

$M = x$

$N = y + ax^2 + ay^2$

$\dfrac{\partial M}{\partial y} = 0 \quad \dfrac{\partial N}{\partial x} = 2ax$

$\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

$\Rightarrow$ The D.E. is not exact.

Problem ③: Solve $(2xy^2 - y)\,dx + x\,dy = 0$.

Solution:

$2xy^2\,dx - y\,dx + x\,dy = 0$.

$\Rightarrow 2x\,dx + \dfrac{x\,dy - y\,dx}{y^2} = 0$.

$\Rightarrow 2x\,dx - d\left(\dfrac{x}{y}\right) = 0$.

Integrating, we get $x^2 - \dfrac{x}{y} = c$.

(or) $x^2y - x = cy$ //

$M\,dx + N\,dy = 0$.

$M = 2xy^2 - y \quad N = x$

$\dfrac{\partial M}{\partial y} = 4xy - 1 \quad \dfrac{\partial N}{\partial x} = 1$

$\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

$\Rightarrow$ The D.E. is not exact.

HW Problem ④: Solve $a(x\,dy + 2y\,dx) = xy\,dy$.

⑤ Solve $(y^2e^x + 2xy)\,dx - x^2\,dy = 0$.

I. F. of a homogeneous equation.

If $M\,dx + N\,dy = 0$ be a homogeneous equation in x and y, then I. F. of $M\,dx + N\,dy = 0$ is

$\dfrac{1}{Mx + Ny}$, provided $Mx + Ny \neq 0$.

Problem ⑥ Solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$.

Solution: The given equation is homogeneous in x and y.

$\therefore$ I.F. $= \dfrac{1}{Mx + Ny} = \dfrac{1}{(x^2y - 2xy^2)x - (x^3 - 3x^2y)y}$

$= \dfrac{1}{x^2y^2}$

Multiplying the given equation by $\frac{1}{x^2y^2}$, the equation becomes

$$\frac{x^2y-2xy^2}{x^2y^2}\,dx-\frac{(x^3-3x^2y)}{x^2y^2}\,dy=0.$$

$$\Rightarrow \left(\frac{1}{y}-\frac{2}{x}\right)dx-\left(\frac{x}{y^2}-\frac{3}{y}\right)dy=0$$

which is exact.

$M=\frac{1}{y}-\frac{2}{x}$, $N=\left(\frac{x}{y^2}-\frac{3}{y}\right)$

$\frac{\partial M}{\partial y}=-\frac{1}{y^2}$, $\frac{\partial N}{\partial x}=\frac{1}{y^2}$.

$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$.

$\Rightarrow$ eqn. is exact.

Integrating, we get the solution

$\int M\,dx+\int$ (those terms of N not containing x) $dy = c$.

$$\Rightarrow \boxed{\frac{x}{y}-2\log x+3\log y=c.}$$ which is the required solution.

(or) $\log\frac{y^3}{x^2}+\frac{x}{y}=c$ //.

Problem (7) Solve $x\,dx=y\,(x^2+y^2-1)\,dy$. (General type).

Solution:

$2x\,dx = 2y\,(x^2+y^2-1)\,dy.$

$2x\,dx+2y\,dy=2y\,(x^2+y^2)\,dy.$

$\frac{2x\,dx+2y\,dy}{x^2+y^2}=2y\,dy.$

$\Rightarrow \frac{d(x^2+y^2)}{x^2+y^2}=2y\,dy.$

Integrating, $\log(x^2+y^2)=y^2+c$ //.

~~Problem (8)~~

Problem ④

Solve $(1-x^2)\frac{dy}{dx} + 2xy = x\sqrt{1-x^2}$ given that $y=0$ when $x=0$.

Solution: Given equation is

$$(1-x^2)\frac{dy}{dx} + 2xy = x\sqrt{1-x^2}.$$

$$\Rightarrow \frac{dy}{dx} + \frac{2x}{1-x^2}y = \frac{x\sqrt{1-x^2}}{1-x^2}.$$

$$\Rightarrow \frac{dy}{dx} + \frac{2x}{1-x^2}y = \frac{x}{\sqrt{1-x^2}} \longrightarrow (1)$$

which is of the standard form $\frac{dy}{dx} + Py = Q$.

Here $P = \frac{2x}{1-x^2}$, $Q = \frac{x}{\sqrt{1-x^2}}$.

Put $1-x^2 = t$
$\Rightarrow -2x\,dx = dt$
$\Rightarrow 2x\,dx = -dt$.

$$\int P\,dx = \int \frac{2x}{1-x^2}dx$$

$$= \int -\frac{dt}{t} = -\log t$$

$$= -\log(1-x^2)$$

$$= \log(1-x^2)^{-1}.$$

$$\Rightarrow e^{\int P dx} = e^{\log(1-x^2)^{-1}} = (1-x^2)^{-1}.$$

$$\Rightarrow \boxed{IF = \frac{1}{1-x^2}}$$

Solution is $y(IF) = \int Q(IF)\,dx + c.$

$$\Rightarrow \frac{y}{1-x^2} = \int \frac{x}{\sqrt{1-x^2}} \cdot \frac{1}{1-x^2}dx + c = \int \frac{x}{(1-x^2)^{3/2}}dx + c,$$

$$\Rightarrow \frac{y}{1-x^2} = \int \frac{x}{\sqrt{1-x^2}\,(1-x^2)}dx + c$$

$$= \int \frac{x}{(1-x^2)^{3/2}}dx + c.$$

Scanned by CamScanner

$$\frac{y}{1-x^2} = \int \frac{x\,dx}{(1-x^2)^{3/2}} + c \rightarrow (2)$$

Put $t = 1-x^2$

$\Rightarrow dt = -2x\,dx$

$-\frac{dt}{2} = -x\,dx$

$$= \int \frac{1}{t^{3/2}}\left(\frac{-dt}{2}\right) + c.$$

$$= -\frac{1}{2}\int t^{-3/2}\,dt + c.$$

$$= \cancel{-\tfrac{1}{2}}\left[\frac{t^{-1/2}}{\cancel{-1/2}}\right] + c.$$

$$= \frac{1}{\sqrt{t}} + c = \frac{1}{\sqrt{1-x^2}} + c.$$

$$\Rightarrow \boxed{\frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} + c} \rightarrow (3)$$

$\therefore$ When $x=0,\ y=0$. $\therefore (2) \Rightarrow 0 = \frac{1}{\sqrt{1}} + c.$

$\Rightarrow c+1=0 \Rightarrow \boxed{c=-1}$

$$\therefore \boxed{\frac{y}{1-x^2} = \frac{1}{\sqrt{1-x^2}} - 1}$$ is the required solution.

<u>Alternate method to solve (2)</u>

Put $x = \sin\theta$

$\Rightarrow dx = \cos\theta\,d\theta$

$1-x^2 = 1-\sin^2\theta$

$= \cos^2\theta$.

$$\int \frac{x\,dx}{(1-x^2)^{3/2}} + c = \int \frac{\sin\theta \cdot \cos\theta\,d\theta}{(\cos^2\theta)^{3/2}} + c$$

$$= \int \frac{\sin\theta\cos\theta\,d\theta}{\cos^3\theta} = \int \frac{\sin\theta}{\cos^2\theta}\,d\theta + c$$

$\frac{\sin\theta}{\cos^2\theta} = \frac{\sin\theta}{\cos\theta}\cdot\frac{1}{\cos\theta}$

$= \tan\theta \cdot \sec\theta$.

$$= \int \tan\theta\sec\theta\,d\theta. = \sec\theta + c.$$

$$= \frac{1}{\sqrt{1-x^2}} + c. \text{ which is } (3)$$

$\cos\theta = \sqrt{1-x^2}$

$\Rightarrow \sec\theta = \frac{1}{\sqrt{1-x^2}}$

Scanned by CamScanner

I.F. for an equation of the type
$f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$.

If the equation $M\,dx + N\,dy = 0$ be of this form, then I.F $= \frac{1}{Mx - Ny}$, provided $Mx - Ny \neq 0$.

Problem (8) solve $(1+xy)\,y\,dx + (1-xy)\,x\,dy = 0$

Solution: The given equation is of the form
$f_1(x,y)\,y\,dx + f_2(x,y)\,x\,dy = 0$.

Here $M = (1+xy)y$; $N = (1-xy)x$.

$$IF = \frac{1}{Mx - Ny} = \frac{1}{(1+xy)xy - (1-xy)xy}$$

$$= \frac{1}{xy + x^2y^2 - xy + x^2y^2} = \frac{1}{2x^2y^2}$$

Multiplying throughout by $\frac{1}{2x^2y^2}$, we get

$$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right)dy = 0,$$

which is an exact equation.

Solution is $\int\limits_{(y\ \text{Constant})} M\,dx + \int (\text{those terms of } N \text{ not Containing } x)\,dy = c$.

$$\Rightarrow \frac{1}{2y}\left(-\frac{1}{x}\right) + \frac{1}{2}\log x - \frac{1}{2}\log y = c$$

$$\Rightarrow \boxed{\log\frac{x}{y} - \frac{1}{xy} = c_1}$$

Problem 9 Solve $y(xy + 2x^2y^2)\,dx + x(xy - x^2y^2)\,dy = 0$.

HW I.F. $= \frac{1}{Mx - Ny} = \frac{1}{3x^2y^3}$.

Ans. $\log\left(\frac{x^2}{y}\right) - \frac{1}{xy} = c$. //

Scanned by CamScanner

In the equation $Mdx + Ndy = 0$

(a) if $\dfrac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}$ is a function of x only, $= f(x)$ (say)

then $IF = e^{\int f(x)dx}$.

(b) If $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}$ is a function of y only $= F(y)$, (say),

then $I.F. = e^{\int F(y)dy}$.

Problem (10) Solve $\left(xy^2 - e^{1/x^3}\right) dx - x^2 y\, dy = 0$.

Solution Here $M = xy^2 - e^{1/x^3}$, $N = -x^2 y$.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2 y} = \frac{4}{x}$$ which is a function of x only,

$$\therefore IF = e^{\int -\frac{4}{x} dx} = e^{-4\log x} = e^{\log x^{-4}}$$

$$\Rightarrow IF = \frac{1}{x^4}.$$

Multiplying throughout by $\frac{1}{x^4}$, we get

$$\left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{1/x^3}\right) dx - \frac{y}{x^2} dy = 0.$$

which is an exact equation.

Solution is $\int \left(\frac{y^2}{x^3} - \frac{e^{1/x^3}}{x^4}\right) dx + 0 = c$,

$$\Rightarrow \frac{-y^2 x^{-2}}{2} + \frac{1}{3}\int e^{x^{-3}} \left(-3x^{-4}\right) dx = c$$

$$\Rightarrow \frac{1}{3} e^{x^{-3}} - \frac{1}{2} y^2/x^2 = c. //$$

Problem 11 Solve $(xy^3+y)dx + 2(x^2y^2+x+y^2)dy = 0$

$M = xy^3+y$, $N = 2(x^2y^2+x+y^2)$.

Here $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{y(xy^2+1)}(4xy^2+2-3xy^2-1)$.

$= \frac{1}{y}$, which is a function of y alone.

$\therefore IF = e^{\int \frac{1}{y}dy} = e^{\log y} = y$

$\Rightarrow \boxed{IF = y}$

Multiplying by y, we get

$(xy^4+y^2)\,dx + (2x^2y^3+2xy+2y^5)\,dy = 0$

which is an exact equation.

$\therefore$ Solution is $\int (xy^4+y^2)\,dx + \int 2y^5\,dy = c$

$\Rightarrow \frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c$ //

HW

(12) Solve $(y-3x^2)\,dx - x(1-xy^2)\,dy = 0$.

(13) Solve $\frac{dy}{dx} = \frac{2x}{x^2+y^2-2y}$.

(14) $(1+xy^2)\,dx + (1+x^2y)\,dy = 0 \longrightarrow$ solve.

(15) Solve $(x^2+y^2)(x\,dx+y\,dy) = a^2(x\,dy - y\,dx)$.

I order Higher Degree D.E

Put $\frac{dy}{dx} = p$.

Type : I — Equations solvable by p.

II — y

III — x

Type I Equations solvable by p.

Procedure (working Rule)

Equation is of the form

$a_1\left(\frac{dy}{dx}\right)^n + a_2\left(\frac{dy}{dx}\right)^{n-1} + \dots + a_n y = 0 \rightarrow (*)$

$\Rightarrow a_1 p^n + a_2 p^{n-1} + \dots + a_n y = 0 \rightarrow (**)$.

Solving (**), we get n values of p.

$p = f_1(x,y)$, $p = f_2(x,y)$, ……, $p = f_n(x,y)$.

$p = f_1(x,y)$ $\Rightarrow$ $\frac{dy}{dx} = f_1(x,y) \rightarrow (1)$

But $\frac{dy}{dx} = p$ $\quad \frac{dy}{dx} = f_2(x,y) \rightarrow (2)$

$\vdots$

$\frac{dy}{dx} = f_n(x,y) \rightarrow (n)$

These (n) equations are of I order I degree.

Solving (n) equations, we get n Independent solutions

$\phi_1(x,y,c_1) = 0$, $\phi_2(x,y,c_2) = 0$, ……, $\phi_n(x,y,c_n) = 0$.

Example: $p = xy$ But $p = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = xy \rightarrow (1)$.

(1) is I order, I degree equation.

Using variable separable method, we can solve (1).

$\Rightarrow \frac{dy}{y} = x\,dx$. Integrating we get

$$\boxed{\log y - x^2/2 = c_1}$$

Problems.

① Solve $\left(\frac{dy}{dx}\right)^2 - 3\frac{dy}{dx} + 2 = 0$ by p.

Solution: We know that $\frac{dy}{dx} = p$.

$\Rightarrow p^2 - 3p + 2 = 0$

$\Rightarrow (p-2)(p-1) = 0$

$\Rightarrow p = 2,\ p = 1.$

$\Rightarrow \frac{dy}{dx} = 2 \rightarrow ①$ $\quad \frac{dy}{dx} = 1 \rightarrow ②$

① & ② are I order I degree equations.

$\Rightarrow dy = 2dx$, $\quad dy = dx$.

Integrating we get

$y = 2x + c_1$, $\quad y = x + c_2$

$\Rightarrow y - 2x - c_1 = 0$ & $y - x - c_2 = 0.$

$\therefore$ General solution is

$$\boxed{(y - 2x - c_1)(y - x - c_2) = 0}$$

② Solve $p^3 - 7p - 6 = 0$.

Solution: $p^3 - 7p - 6 = 0$

6
1 2 3

$p = -1 \Rightarrow$

$(-1)^3 - 7(-1) - 6$

$= -1 + 7 - 6 = 0.$

-1	1	0	-7	-6
	0	-1	1	6
	1	-1	-6	0

$\Rightarrow p = -1,\ p^2 - p - 6 = 0.$

$\Rightarrow p = -1,\ (p-3)(p+2) = 0.$

$\Rightarrow p = -1, -2, 3.$ But $p = \frac{dy}{dx}$.

$\Rightarrow \frac{dy}{dx} = -1,\ \frac{dy}{dx} = -2,\ \frac{dy}{dx} = 3.$

Solving, we get

$(y + x - c_1)(y + 2x - c_2)(y - 3x - c_3) = 0.$

③ $2p^2 - (x + 2y^2)p + xy^2 = 0.$

Solution:

Given equation is

$2p^2 - (x+2y^2)p + xy^2 = 0 \rightarrow ①$

$a = 2,\ b = -(x + 2y^2),\ c = xy^2.$

$p = \frac{(x+2y^2) \pm \sqrt{(x+2y^2)^2 - 8xy^2}}{4}$

$= \frac{(x+2y^2) \pm \sqrt{x^2 + 4y^4 + 4xy^2 - 8xy^2}}{4}$

$= \frac{(x+2y^2) \pm \sqrt{x^2 + 4y^4 - 4xy^2}}{4}$

$= \frac{(x+2y^2) \pm \sqrt{(x - 2y^2)^2}}{4}$

$= \frac{(x+2y^2) \pm (x - 2y^2)}{4}$

$= \frac{x + 2y^2 + x - 2y^2}{4},\ \frac{x + 2y^2 - x + 2y^2}{4}$

$= \frac{2x}{4},\ \frac{4y^2}{4} = \frac{x}{2},\ y^2.$

$\Rightarrow \frac{dy}{dx} = \frac{x}{2},\ \frac{dy}{dx} = y^2.$

$\Rightarrow dy = \frac{x}{2}dx,\ \frac{dy}{y^2} = dx.$

Integrating we get

$y = \frac{x^2}{4} + c_1,\ \frac{y^{-1}}{-1} = x + c_2.$

General solution is

$\left(y - \frac{1}{4}x^2 - c_1\right)\left(\frac{1}{y} + x + c_2\right) = 0.$

$\Rightarrow (4y - x^2 - 4c_1)(xy + yc_2 + 1) = 0$